X Resolving the antinomies

Firstly, the difference between antinomies and paradoxes, which is not clearly defined in the literature, is defined. Antinomies are contradictions that cannot be (or appear to be) resolved. The paradox is only apparently contradictory and can be explained logically.
In fact, there are no antinomies in nature, only contradictory attempts by man to explain them. The reason for the apparent contradiction can be uncovered and eliminated, only paradoxes exist. Antinomies are apparent logical errors of thought. In the following, only the 'antinomies' that have played or still play a decisive role in mathematics will be discussed.

1 Russell's antinomy
2. the barber
3. Burali-Forti
4. The set of all sets
5 Epimenides the Liar

1 Russell's antinomy

Russell's antinomy, which is not primarily aimed at the transfinite, but questions the core of set theory, the concept of set, was perceived as so serious that it initiated the so-called foundational crisis of mathematics at the beginning of the 20th century.
Russell distinguishes between sets that contain themselves, such as the set of all sets, and the usual sets, such as the set of all apples, which do not contain themselves.
The bone of contention is the set R, the set of all sets that do not contain themselves.
Russell asks whether R contains itself or not and states that a contradiction can be proved in both cases.
If R does not contain itself, this contradicts the premise that all sets that do not contain themselves should be summarised. If R contains itself, this violates the condition that only sets that do not contain themselves should be considered. If 'being an element' is expressed by the sign $\in$ and 'not being an element' by $\notin$, Russell's antinomy is formulated as follows:

$$\mathbf{R \in R \leftrightarrow R \notin R}$$

From the premise that R is an element of R, it follows contradictorily that R is not an element of R and vice versa.
The crucial question is whether there can even be sets that contain themselves.
For example, consider the set of all horses. If it contained itself, then all horses would be counted a second time. The horses would be counted twice and this contradicts the definition of the concept of a set. This argument applies to any set of any elements. **Sets cannot contain themselves.**
Russell's set R cannot contain itself either. The prerequisite of the antinomy proves to be invalid.
$R \in R$ is false, but $R \notin R$ is true. The equivalence $R \in R \leftrightarrow R \notin R$ does not exist, the contradiction is resolved.
Mathematicians did not and still do not see the problem with $R \in R$, but erroneously with R. "There is no set of all sets that does not contain itself as an element" applies in all modern axiomatic set theories. According to mathematicians, Russel's set is not an admissible set, but an "antinomic" set. The axioms of ZFC and other axioms were unnecessarily formulated in such a way that R is inadmissible..

2. The barber

In addition to the antinomy named after him, Russell invented the 'barber' to illustrate his antinomy metaphorically.
(1) "One can define a barber as one who shaves all those and only those who do not shave themselves."

The question is: does the barber shave himself?
If one assumes that he shaves himself, this contradicts the claim that he shaves all those who do not shave themselves. If it is assumed that he does not shave himself, this is also incompatible with the requirement that he shave all those who do not shave themselves.
There is a simple solution to the question of whether the barber shaves himself or not: you have to observe him, then you know.
- If the barber does not shave himself, he is one of those who do not shave themselves.
- If the barber shaves himself, he does not belong to all those who do not shave themselves.
There is no inconsistency in reality, it must be hidden in the (ill)logical construction of the 'barber'.
The barber who shaves all those who do not shave themselves does not exist if he does not shave himself.
The barber who shaves all those who do not shave themselves also does not exist if he shaves himself.
(2) The barber according to definition (1) does not exist.
Russel's antinomy is also characterised by non-existence, $R \in R$ does not exist.

3. Burali-Forti

The Burali-Forti antinomy was the first antinomy of set theory to be established.
The fact that the infinite sequence of Cantor's ordinal numbers, including the transfinite ones, cannot be summarised as a set was seen as an antinomy, since supposedly only sets exist.
Even within Cantor's thought structure, the term "antinomic" is revealing. Cantor himself accepted the existence of this "immensity". However, due to the fixation on sets, an antinomy was asserted.
In fact, there is no set but an infinite sequence, which can in no way be seen as an antinom.
The axiom system ZFC that is predominantly used today assumes, contrary to the truth, that only sets exist, i.e. that there are only limited combinations of elements. In the meantime, however, there are axiom systems such as those of v. Neuman, Bernays, Gödel, NBG, which introduce unbounded infinite sequences as so-called "real classes".

4. The set of all sets

The most comprehensive set conceivable should be the "set of all sets". But then it can immediately be objected that the "set of all subsets" is larger. This fact is known as "Cantor's antinomy".
The set of subsets contains the same elements as the set, only in the subsets of these elements are combined in different summaries. For example, a set of 3 horses has the following subsets: The first, the second and the third horse. The first and the second, the first and the third, the second and the third and all three horses. That is seven subsets. A set of n elements then has 2n-1 subsets. This is one less than the previous theory requires, as the non-existent empty set is incorrectly counted.
There is nothing antinomic about the statement that the subsets are larger than the underlying sets. The fact that the correspondingly larger set of subsets follows from the set of all sets cannot be seen as antinomic either. However, there is no limited "set of all sets", but an unlimited "sequence of all sequences".

5 Epimenides the liar

In his 1931 publication, Gödel cited the 'liar' as an "epistemological antinomy", which corresponds to his 1st incompleteness theorem on the linguistic logical level.
Epimenides, the Cretan, says: "All Cretans lie".
If Epimenides is lying, his statement that all Cretans lie would be true. That is contradictory. If he is telling the truth, he would be confirming that he is lying with his statement. That is also a contradiction. It is not possible to decide whether the Cretan is telling the truth or whether he is

lying. The attempt to decide the "Cretan" leads to antinomian statements.
But why is it not possible to decide whether the truth is being told or whether it is a lie?
There is no information as to which lie the Cretans are supposedly spreading**. The creeter's sentence contains a hidden variable, the 'lie'. If this is specified, the variable bound, it can also be decided whether all Cretans are spreading it or not.** "All Cretans lie about the Athenians" would be such an example. It is now possible to check whether this statement is true or not. Previously, the necessary information was missing to decide whether Epimenides' sentence is true or not.
Of course, not all linguistic sentences that contain a hidden variable such as 'lie' are undecidable. "Socrates says: 'All Cretans lie' is decidable. **Only the combination with self-reference implies undecidability in analogy to Gödel's theorem.**