II Inconsistence of the transfinite number ω and the set ℕ

April 2023. For more than 2 millennia, Aristotle's "infinitum actu non datur", "there is no limited infinity", dominated the perception of the infinite. End of the 19th century Georg Cantor introduced the actual infinity or transfinite with the postulate of limits, even levels in infinity, with his set theory. The result was a foundational crisis of mathematics[1]
Cantor's doctrine was not axiomatically founded. The axiomatic system of Zermelo and Fraenkel then formally legitimized the transfinite. The crisis is considered to be overcome thereby.
Cantor's transfinite was initially highly controversial, but ultimately prevailed. Today, in its axiomatic form, it constitutes one of the foundations of mathematics. In the following it will be shown that both, the axiomatic and Cantor's justification of the transfinite number ω and the set ℕ of the natural numbers, are contradictory. The foundational crisis persists. Reintroducing the true meaning of 0, "nothing" and "no number", refutes the existence of the transfinite terms.

1. Introduction

1.1. Cantor

Before Cantor, Aristotle's "infinitum actu non datur", "there is no actual[2] infinity", applied. For more than two millennia, this sentence established the view of infinity. For example, the sequence of natural numbers continues indefinitely:
(1) 1 2 3 4 …..........n …..
On the other hand, in the 1970s, **Georg Cantor demanded limits, even levels, in infinity in his set theory.** He describes them as "actually infinite" or "transfinite". The infinite sequence (1) never breaks off, but Cantor beyond demands a larger transfinite limit ω (2).The term "transfinite" is explained by the fact that the natural numbers (1) for Cantor are only potentially infinite, i.e. ultimately finite, only ω is transfinite.
The postulate of a transfinite limit of never ending numbers requires an intuitively understandable explanation. According to Cantor, the limit value of irrational numbers, such as √2 (3), is "equal in its innermost essence to the transfinite numbers". The mapping of the infinite sequence of approximate values and √2 to natural numbers and ω is supposed to justify the existence of the transfinite number.

(2)	1	2	3	4	5	6	…................ n …..................	ω
(3)	1	1.4	1.41	1.414	1.4142	1.41421	…...............….......	√2

Cantor's works on set theory, that contains not only transfinite limits but also levels, antinomies and counter-intuitive propositions, was initially perceived highly controversial. **Set theory caused a fundamental crisis of mathematics that lasted for decades.** Cantor's teacher Leopold Kronecker,

1 The topic including the bibliographical references are part of the book by Gert Treiber, " Nichts", Krise und reEvolution der Grundlagen der Mathematik, Cuvillier Verlag 2020.
2 i.e. limited

for example, judged that he did not know whether the transfinite was philosophy or religion, but it was certainly not mathematics. **Henri Poincaré criticized that future generations will regard the transfinite as a disease from which mathematics has recovered.** Richard Dedekind, on the other hand, was one of Cantor's steadfast supporters. **Ultimately, David Hilbert was decisive for the breakthrough: "No one should be able to drive us out of the paradise, Cantor created for us."**

1.2. The axiomatic system ZFC

In order to secure the foundations of set theory, Ernst Zermelo and Abraham Fraenkel developed the axiomatic system ZFC, that allowed to deduce the theorems of set theory without contradiction. It is considered the end point of the crisis. Today, only a small minority criticizes Cantor and ZFC respectively, without being able to prove contradictions. **The following explanations demonstrate that there are contradictions, the foundational crisis of mathematics still exists.** Another article[3] reveals serious mistakes made by Cantor.

In the article on hand, the refutation of the transfinite is limited to the transfinite set ℕ and number ω. The inconsistency of the ZFC axiom of infinity for transfinite sets in general, given the empty set, is the subject of a further article[4]

2. The justification of ω and ℕ

2.1. Cantor's rationale

Cantor first spoke of the transfinite as the "foundation of the real existing in nature". He saw the "body matter" represented by the transfinite set of natural numbers. Later he justified transfinite numbers only as **"conceivable and not objectionable".**

His **explanation by the mapping of irrational numbers** excluded the 0 as an element (2).
Cantor rightly did not see 0 as a natural number.

2.2. Justification by ZFC

2.2.1 Intuitive reasoning

Fraenkel demanded the natural number 0 "with imperative necessity", "since this is the only way the transfinite number ω can be justified".

The decisive reason is already evident in the finite. The set n +1 of the numbers 0, 1, 2, 3, 4, ….... n is larger than the last number n:

(4){ 0, 1, 2, 3, 4, ….......... n } = n + 1

In addition, the number n + 1 delimits the sequence of the numbers 0, 1, 2, 3, 4, ….. n.

Although there is no last number in the infinite sequence of natural numbers, it is understandable from (4) that the transfinite set ℕ of the natural numbers according to ZFC is larger than the natural numbers (5), and that these numbers are limited by a number ω (6).

(5) { 0, 1, 2, 3, 4, …........n, …...... } = ℕ

(6) 0, 1, 2, 3, 4, …...........n, ….................ω

ω is followed by further transfinite numbers ω + 1, ω + 2, …....... .

The "number 0", that increases the set, is an indispensable prerequisite for ℕ and ω in ZFC.

2.2.2. Formal justification by the axiom of infinity

Formally, the axiom of infinity demonstrates the postulate to include 0 in the transfinite set. Applied to the set ℕ of the natural numbers x, the axiom is formulated by (7):

(7) $\exists \mathbb{N}: 0 \in \mathbb{N} \wedge \forall x: (x \in \mathbb{N} \rightarrow x \cup \{ x \} \in \mathbb{N})$

The existence of the transfinite set ℕ premises that 0 is the first element of ℕ and that for all numbers x applies: If x is an element of ℕ, then x united with { x } is also an element of ℕ. Fraenkel's imperative necessity is thus formally established. Without going into detail, we note that x ∪ { x } guarantees that the next larger number, including ω, is always generated.

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3 Article III Cantor's Errors in the Theory of the Transfinite

4 Article IV Inconsistency of the empty and the transfinite Set

3. The refutation of ω and ℕ

3.1. Intuitive rebuttal

The 0 is, empirically proven and secured by a new axiom, not a number, but "nothing" of the numbers, i.e. "no number".[5] As a result, a set of numbers cannot be enlarged thereby.
In the case of sums, 0 is already the neutral element; it does not increase the sum, as is shown by (8) examplarily. In contrast, according to current theory, the set is increased by the 0 (9).

(8) Sum: $(0 + 1_1 + 1_2 + \ldots\ldots\ldots\ldots + 1_n) = n$

(9) Present set theory: $\{ 0,\ 1_1,\ 1_2, \ldots\ldots\ldots\ldots,\ 1_n \} = n + 1$

Because of the meaning "nothing", "no number" of 0, set (10) and sum (8) are equal. This resolves a discrepancy between arithmetic and set theory.

(10) Revised set theory: $\{ 0,\ 1_1,\ 1_2, \ldots\ldots\ldots\ldots,\ 1_n \} = n$

The set is not larger than the last number n, as incorrectly required by (4). This holds for all natural numbers, as can be proved by mathematical induction. Including the 0 doesn't change anything.
The transfinite terms ℕ and ω that are larger than natural numbers do not exist.
The following considerations confirm these statements.

3.2. Empirical refutation by units of length and the 0-point

The sequence of the same length units of a straight line (11) is ordered by the natural numbers, the 0-point is assigned to the 0 (12). The set as well as the sum of units of length l_i (i = 1, 2, … n …) corresponds to the entire distance:

(11) |—l_1—|—l_2—|—........................—l_n—| ….................................

(12) 0—l_1—|—l_2—|—........................—l_n—| ….................................

The sum as well as the set of length units, including the 0-point, is not greater than that specified by the element l_n. This reasoning is also true at infinity due to mathematical induction:
A greater length that would represent a transfinite set of units does not exist.
The mapping of units of length and the 0-point to natural numbers and the 0 confirm that the transfinite set ℕ and the number ω do not exist.

3.3. Mathematical induction refutes Cantor

Cantor rightly did not see 0 as a natural number. The set of natural numbers is always equal to the last number n and never larger than a natural number:

(13) { 1, 2, 3, 4, ….......... n } = n

Mathematical induction rules out a transfinite set ℕ that would be larger than the natural numbers and refutes Cantor.
Fraenkel also contradicts Cantor by demanding 0 as a natural number “with imperative necessity” in order to justify the transfinite number ω. However, he insinuates agreement with Cantor by misquoting him. Cantor published the sequence 1, 2, ,3, ... n ….. ω, ω+1, …. without the 0, Fraenkel presumes the sequence 0, 1, 2, 3, ….. n ….. ω, ω+1, ….. with the 0.

3.4. Formal refutation of the ZFC-axiom of infinity

In article I, the axiom of the "nothing" of numbers is established:

(14) $\exists \mathcal{S}: \mathcal{S} = 0 \leftrightarrow \forall x: \neg (\exists x : x = 0)$

The sign $\mathcal{S} = 0$, representing "nothing", "no number", exists. This proposition is equivalent to stating for all numbers x: a number x that equals 0 does not exist.
Axiom (14) excludes the premise of (7) .
The ZFC-axiom of infinity is refuted by the axiom of "nothing" of numbers.
The existence of the "nothing" of numbers proves non-existence of the transfinite set ℕ.

4. Epistemology

5 Article I Zero is not a number but represents “nothing”

Another fundamental objection to the existence of the transfinite numbers is of epistemological nature. **David Hilbert, one of the most renowned mathematicians of his time, has admitted that the transfinite contradicts reality and is just an idea.** He has been very clear on this in 1926 in *Über das Unendliche* [20]. “The infinite is nowhere realized; it does not exist in nature, nor is it permissible as a basis for our intellectual thinking”. The transfinite however is an “idea secured independently of all logic”. Hilbert thus confirms a "successful, complete calculus" for Cantor's doctrine, free of contradictions. He also approved ZFC. **Hilbert notes a contradiction to reality, nevertheless insists the mathematical theory to be consistent. But epistemology demands, that an idea, that contradicts reality, is necessarily wrong.**
Also today, no mathematician or logician will claim that limits and levels in infinity can be brought in agreement with physics and astronomy without contradiction. Nevertheless, a mathematically consistent theory of the transfinite is still postulated rationalistically, although it contradicts empiricism. The philosopher Immanuel Kant has overcome such contrariness by his transcendental philosophy, contradictions between rationalistic ideas and empirical facts are excluded. The transfinite already fails because of Kant's epistemology.